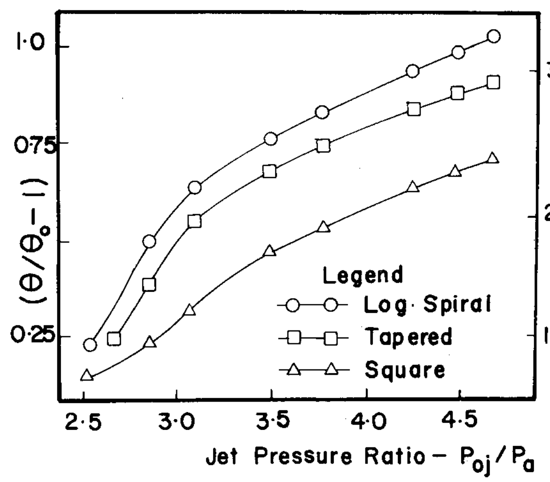

Fig. 3 End-wall temperatures.

Nevertheless, the present results are useful from a comparative rather than absolute standpoint.

Tests were also conducted to determine those locations of the tube mouth relative to the nozzle that produced maximum end-wall temperatures. Results for the logarithmic spiral profile at the jet pressure ratio of 3.77 show that the maximum temperature varies periodically with the nozzle-tube separation h due to the periodic structure of the exciting jet. The highest temperatures were recorded at $h/d = 1.7$. This means that the peak corresponds to a location where the tube mouth is situated downstream of the first pressure cell. This result differs from the findings of Przirembel and Fletcher.[8] However this difference might be explained by the fact that in their case, the open-end walls were rounded in contrast to the 30-deg chamfered entrances of the present tests.

Conclusion

Thermocouple measurements have demonstrated that a Hartmann-Sprenger tube with a logarithmic spiral profile achieves a significantly greater base temperature than a 10-deg tapered tube or a rectangular tube of comparable length. This improved performance is interpreted to result from a higher degree of shock amplification which produces a greater entropy rise in the shock-heated indigenous gas that remains trapped inside the tube. Therefore, it appears that an axisymmetric logarithmic spiral tube could be used to improve the performance further.

Acknowledgments

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Proper Definition of Curvature in Nonlinear Beam Kinematics

Dewey H. Hodges*
U.S. Army Research and Technology Laboratories (AVSCOM)
Ames Research Center, Moffett Field, California

Introduction

IN the analysis of large deflections of beams, it is important to include effects of geometric nonlinearity. Sometimes the nonlinear curvature expressions used in published works are incorrect and the reasons why may appear subtle. The purpose of this Note is to show why the curvature expression in the typical calculus text is not always appropriate for the kinematics of a deformed beam. To illustrate this, a relatively simple example involving the kinematics of a beam deforming in a plane is used. No attempt is made here to cite all the references pertinent to this subject. Neither is there an attempt to present a new development; the main motivation is tutorial.

Development of Equations

The development herein is similar to a more general one found in Ref. 1. Consider a beam with coordinate systems, as shown in Fig. 1. For purposes of discussion, the neutral axis of the undeformed beam lies along the x axis and the y axis is normal to the x axis at the root end of the beam denoted by E. The x and y axes are fixed in space. The position of a material point M relative to E in the beam prior to deformation is given by

$$\underline{R}^{ME}|_{\text{undeformed}} = x\underline{e}_x + y\underline{e}_y \tag{1}$$

where $\underline{e}_x$ and $\underline{e}_y$ are unit vectors along the x and y axes, respectively. It is assumed that a set of material points at $x = \text{const}$, in a plane prior to deformation, remain in a plane during deformation. Furthermore, the cross section is assumed not to deform in its plane. Then, as usual for a Lagrangian procedure, the beam's geometry during deformation can be characterized by the unknown functions $u(x)$, $v(x)$, and $\zeta(x)$ so that the same material point M has location relative to E during deformation given by

$$\underline{R}^{ME} = (x+u)\underline{e}_x + v\underline{e}_y + y\underline{e}_\eta \tag{2}$$

where $\underline{e}_\eta$ is a unit vector in the plane of the cross section during deformation. Another important unit vector is $\underline{e}_\xi$,

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*Research Scientist and Theoretical Group Leader, Rotorcraft Dynamics Division, Aeromechanics Laboratory, Associate Fellow, AIAA.

which remains normal to the cross section during deformation. The unit vectors $\underline{e}_\xi$ and $\underline{e}_\eta$ lie along the ξ and η axes, respectively, obtained from rotation of the x and y axes through the angle ζ, as shown in Fig. 1.

$$\underline{e}_\xi = \underline{e}_x \cos\zeta + \underline{e}_y \sin\zeta \tag{3}$$

$$\underline{e}_\eta = -\underline{e}_x \sin\zeta + \underline{e}_y \cos\zeta \tag{4}$$

Note that when $u(x)$, $v(x)$, and $\zeta(x)$ vanish for all x, M is coincident with its position prior to deformation. Also note that the coordinate x is a measure of the distance of M from the root end of the undeformed beam. Similarly, y is the distance of M from the neutral axis of the undeformed beam and remains constant during deformation.

The unit vector tangent to the neutral axis at a point N (for which $y=0$) during deformation is given by the derivative of $\underline{R}^{NE}$ with respect to s, the distance along the deformed beam neutral axis.

$$\frac{d\underline{R}^{NE}}{ds} = (x+u)^+ \underline{e}_x + v^+ \underline{e}_y \tag{5}$$

where derivatives with respect to s are denoted by $(\ )^+$. If the tangent is required to be coincident with $\underline{e}_\xi$, then deformation due to transverse shearing is suppressed. This leads to the possibility of elimination of ζ since, by virtue of Eqs. (3) and (5),

$$\cos\zeta = (x+u)^+ \tag{6}$$

$$\sin\zeta = v^+ \tag{7}$$

Equations (6) and (7) can be combined to yield the following relationship:

$$(x+u)^{+^2} + v^{+^2} = 1 \tag{8}$$

It is desirable to express this relationship in terms of derivatives with respect to x, denoted by $(\ )'$. Note that $(\ )^+$ becomes $(\ )'/s'$. Now, multiplication of Eq. (8) through by s'^2 yields

$$(1+u')^2 + v'^2 = s'^2 \tag{9}$$

The elongation of the neutral axis, for small strains and large rotations, can be expressed[2,3] from the derivative of $\underline{R}^{NE}$ and $\underline{e}_\xi$.

$$\epsilon = \underline{e}_\xi \cdot \frac{d\underline{R}^{NE}}{dx} - 1 = s' - 1 \tag{10}$$

The curvature is

$$\kappa = \underline{e}_\eta \cdot \underline{e}_\xi^+ = \zeta^+ \tag{11}$$

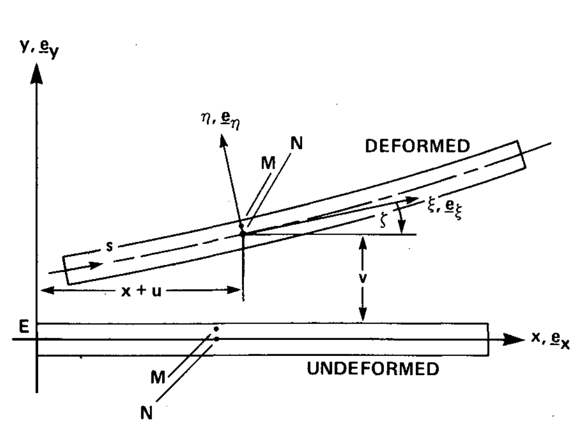

Fig. 1 Schematic of undeformed and deformed beam with pertinent coordinate systems.

In order to eliminate ζ, κ needs to be expressed in terms of u and v. This can be done from Eqs. (6) and (7) in a variety of ways. For example, divide the left and right sides of Eq. (7) by the left and right sides of Eq. (6), respectively. Differentiate both sides with respect to s and solve for the curvature to obtain

$$\kappa = [(1+u')v'' - v'u'']/s'^3 \tag{12}$$

Another approach is to differentiate both sides of Eq. (7) with respect to s and divide the left and right sides by the left and right sides, respectively, of

$$\cos\zeta = \sqrt{1 - v'^2/s'^2} \tag{13}$$

This leads to another expression for the curvature

$$\kappa = \frac{v'' - v's''/s'}{s'\sqrt{s'^2 - v'^2}} \tag{14}$$

These expressions for κ can be simplified further for small strains. By virtue of Eq. (10),

$$s' = 1 + \epsilon \tag{15}$$

It is assumed that Hooke's law applies and thus ϵ must be small compared to unity. Substitution of Eq. (15) into each of Eqs. (12) and (14) followed by expansion of the result in terms of the neutral axis strain ϵ yields simplified expressions for κ. When ϵ is negligible compared to unity, Eq. (12) becomes

$$\kappa = (1+u')v'' - v'u'' \tag{16}$$

Equation (14), with the additional assumption that $\epsilon'v'$ is small relative to v'', becomes

$$\kappa = v''/\sqrt{1-v'^2} \tag{17}$$

Discussion

For a beam with kinematics such as described herein, the bending moment would be proportional to κ. Thus, κ is the *physical* curvature of the deformed beam elastic line. Epstein and Murray[4] identified Eq. (16) as a small-strain approximation that is desirable because it is without quotients or irrational functions of u and v. Hodges[5] has used Eq. (17) because it is a function of v only. These expressions for κ are exact for an inextensional beam, as long as u is properly specified in Eq. (16), and quite accurate for any elastic beam undergoing planar deflection since the axial strain ϵ in an *elastic* beam is of necessity small compared to unity. Furthermore, neither Eq. (16) nor (17) contains any restriction on the magnitude of rotations.[2-4] The terms dropped in the assumption that ϵ is small compared to unity give rise to terms in the axial strain that are nonlinear in elongations. The statement of Venkatesan and Nagaraj[6] that this sort of approximation invalidates the strain for cases other than inextensional is clearly incorrect. This simplification has essentially no effect on the accuracy of the curvature as long as the strains are small, which they must certainly be for application of Hooke's law.

A more familiar expression for curvature, as found in elementary calculus texts, is sometimes mistaken as the physical beam curvature. In some references, for example, Refs. 7-10, the curvature $\bar{\kappa}$ of the curve v vs x is assumed in an analysis to be, or at least stated as, the appropriate nonlinear curvature for a beam where

$$\bar{\kappa} = v''/(1+v'^2)^{3/2} \tag{18}$$

The coefficient of the first nonlinear term of the Taylor series expansion of Eq. (18) has the opposite sign and an exaggerated magnitude relative to that of Eq. (17). For kinematically nonlinear analysis of beams this effect certainly would be noticeable, especially for large deflection problems. It should be obvious that $\bar{\kappa}$ has *no* physical meaning for the problem of the bending of a beam in terms of the coordinate x as defined here. The reason for this is that the coordinate x is along the original position of the beam, as is typical in mechanics. With the present definition of x, Eq. (18) does not account for the well-known effective shortening due to transverse deflections. This effective shortening generates a deflection u even in an inextensional beam, due to a transverse deflection v. If the derivatives in Eq. (18) are redefined to be with respect to the shortened axis $\bar{x}=x+u$, then it is not difficult to show that

$$\kappa=\frac{\mathrm{d}^2 v/\mathrm{d}\bar{x}^2}{[1+(\mathrm{d}v/\mathrm{d}\bar{x})^2]^{3/2}} \tag{19}$$

Thus, the key factor in making the familiar expression for curvature compatible with the kinematics of planar beam deformation is that the independent variable in such an expression must be the actual distance of the material point from the root of the beam (i.e., $\bar{x}$). It should be noted that description of beam kinematics in terms of $\bar{x}$ is often inconvenient since the limits of integration over the beam are functions of the deformation. This is obviously not the case when x is used as the coordinate.

Concluding Remarks

In this Note, a simplified example was chosen deliberately to show that the physical curvature for planar beam deformation is not the same as the commonly known expression found in elementary calculus texts. Even this simplified example proves to be complex relative to linear beam analyses. That pitfalls exist even in the best attempts to adapt textbook expressions to fit one's particular analytical needs should be obvious from this development. Clearly, the safest approach for any nonlinear analysis, including one involving geometric nonlinearity, should be a careful one based on a sound application of fundamental principles.

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Application of the Generalized Inverse in Structural System Identification

Shyi-Yaung Chen* and Jon-Shen Fuh*
Kaman Aerospace Corporation,
Bloomfield, Connecticut

A PRINCIPAL objective of system identification is to improve the mathematical model of a test structure that can then be used to predict the system dynamics and the effects of the modification of the structure. Analytically, the structure is discretized and the mass matrix M_a and stiffness matrix K_a can be obtained using the finite element method. Of course, these matrices are only approximate expressions to the structure. On the other hand, vibration testing provides incomplete information on the modal properties of the actual structure. Two mathematical methods have been applied to the improvement of analytical models. Assuming the measured natural frequencies and mode shapes to be exact, Berman[1] and Berman and Wei[2] used the Lagrange multiplier method to minimize a matrix norm introduced by Baruch and Bar Itzhack.[3] The improved mass and stiffness matrices thus obtained exactly predict the measured natural frequencies and mode shapes. Alternatively, the generalized inverse method was used by Rodden[4] in computing the stiffness matrix, by Berman and Flannelly[5] in identifying the mass matrix, and by Chen and Garba[6] in the parameter estimation. However, these three papers dealt only with linear equations without constraints. In this Note, application of the generalized inverse to constrained systems is presented. The basic assumption is that, as in Refs. 1 and 2, the measured modal properties are exact. It is found that the present approach is physically sound and algebraically simpler in formulation.

Generalized Inverse

Penrose[7] showed that for any matrix $A(p\times q)$ there is a unique matrix $A^+(q\times p)$ satisfying the four equations

$$AA^+A=A,\quad A^+AA^+=A^+,\quad (AA^+)^*=AA^+,$$
$$(A^+A)^*=A^+A \tag{1}$$

where the asterisk denotes the conjugate transpose of a matrix and A^+ is known as the Moore-Penrose generalized inverse. Specifically, if A is real, so is A^+; if A is nonsingular, then $A^+=A^{-1}$. Note also that, if A is of rank q, the A^+ is the only left inverse of A having rows in the row space of A^*. In this case, $A^+=(A^*A)^{-1}A^*$. Conversely, if A is of rank p, then A^+ is the only right inverse of A having columns in the column space of A^* and $A^+=A^*(AA^*)^{-1}$. (See Ref. 8.)

The generalized inverse techniques have been used extensively in solving the linear equation

$$AX=B \tag{2}$$

and the general solution reads

$$X=A^+B+(I-A^+A)Y \tag{3}$$

in which Y is arbitrary. It is obvious that $(I-A^+A)Y$ satisfies the associated equation $AX=0$ and is referred to as the homogeneous solution. The particular solution A^+B is the minimum-norm least-square solution of Eq. (2). That is,

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*Research Engineer Specialist.